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An Improvement of Kalandiya's Theorem

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Kalandiya's theorem in approximation theory [2, 3] for Hölder-continuous functions $f(x)$ of order α ($f \in H_\alpha$) states:

THEOREM 1. *Let a function $f(x)$ of the class H_α be given. For every natural n let $p_n(x)$ be an algebraic polynomial of degree n for which*

$$|f(x) - p_n(x)| \leqslant A_1 n^{-\alpha}, \qquad x \in [-1, 1], \tag{1}$$

where A_1 is a constant. Then one has the estimate

$$\max_{x_1, x_2 \in [-1,1]} \frac{|r_n(x_2) - r_n(x_1)|}{|x_2 - x_1|^\beta} \leqslant A_2 n^{-\alpha + 2\beta}, \tag{2}$$

where $r_n(x) = f(x) - p_n(x)$, β being a positive number such that $2\beta < \alpha$ and A_2 is a constant depending on α and β.

This theorem was recently extensively used for proofs of convergence theorems for quadrature rules for Cauchy type principal value integrals and for the numerical solution of singular integral equations. Six references, by several authors, are reported in [1], where a new proof of this theorem was also given.

We will show the following improvement of Kalandiya's theorem:

THEOREM 2. *Let a function $f(x)$ of the class H_α $(0 < \alpha \leqslant 1)$ on $[-1, 1]$ be given. Then there exists a sequence of polynomials $p_n(x)$ of degree n for which*

$$\max_{x_1, x_2 \in [-1,1]} \frac{|r_n(x_2) - r_n(x_1)|}{|x_2 - x_1|^\beta} \leqslant A_3 n^{-\alpha + \beta}, \tag{3}$$

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where $0<\beta<\alpha$ *and* A_3 *is a constant depending on* f, α *and* β, *but independent of* n.

Proof. (In the sequel we will denote by A_i positive constants independent of n and x.) The method of proof is completely analogous to that presented in [1], but with the following difference: we substitute $x=ct$ $(c>1)$ in (3). Then we have to show that

$$\frac{|r_n^*(t_2)-r_n^*(t_1)|}{|t_2-t_1|^\beta}\leqslant A_4 n^{-\alpha+\beta}, \qquad t_1, t_2\in[-1/c, 1/c], \tag{4}$$

where $r_n^*(t)$ is given by $r_n^*(t)=f^*(t)-p_n^*(t)$, $f^*(t)$ is defined in such a way that it coincides with $f(ct)$ for $t\in[-1/c,1/c]$ and belongs to H_α for $t\in[-1,1]$, and $p_n^*(t)$ denotes a sequence of polynomials associated to $f^*(t)$ along $[-1,1]$ so that

$$|r_n^*(t)|\leqslant A_5 n^{-\alpha}. \tag{5}$$

(The existence of this sequence is assured by Jackson's theorem [4].) Concerning f^*, one may take, for example,

$$\begin{aligned} f^*(t)&=f(-1), && t\in[-1,-1/c],\\ &=f(ct), && t\in[-1/c,1/c],\\ &=f(1), && t\in[1/c,1]. \end{aligned}$$

Using the inequality (cf. [5]),

$$|p_n^{*\prime}(t)|\leqslant\frac{n}{2(1-t^2)^{1/2}}\,\omega\left(2\sin\frac{\pi}{2n}, p_n^*\right), \qquad t\in[-1,1]$$

(ω denoting a modulus of continuity), we find since $f^*\in H_\alpha$

$$|p_n^{*\prime}(t)|\leqslant A_6 n^{1-\alpha}, \qquad t\in[-1/c,1/c]. \tag{6}$$

Proceeding now analogously to [1] we can deduce the desired inequality (4) for $|t_2-t_1|<1/n$ from

$$\frac{|r_n^*(t_2)-r_n^*(t_1)|}{|t_2-t_1|^\beta}\leqslant\frac{|f^*(t_2)-f^*(t_1)|}{|t_2-t_1|^\beta} + \left|\frac{p_n^*(t_2)-p_n^*(t_1)}{t_2-t_1}\right||t_2-t_1|^{1-\beta}$$

by taking into account that $f^*\in H_\alpha$, the mean value theorem and (6), whereas the case $|t_2-t_1|\geqslant 1/n$ follows from (5). This completes the proof. ∎

It should be mentioned that Theorem 1 holds for every sequence of polynomials $\{p_n\}$ satisfying (1), whereas in Theorem 2 one has to choose a particular sequence $\{p_n^*\}$. Indeed, there exist sequences $\{p_n\}$ satisfying (1) for which the order $\mathcal{O}(n^{-\alpha+2\beta})$ is best possible.

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